Noise-induced phase transition in soft Ising spins with a fluctuating interaction

Seunghwan Kim,* Seon Hee Park, and C. S. Ryu
Research Department, Electronics and Telecommunications Research Institute P.O. Box 106, Yusong-gu, Taejon 305-600, Korea
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We investigate noise-induced phase transitions in soft Ising spins, which have continuous values, with a fluctuating interaction within the framework of the mean-field scheme. The fluctuating interaction provides an effective ferromagnetic coupling, leading to a ferromagnetic order. The interplay of the fluctuating interaction and an additive noise leads to a reentrant transition, triple transitions, and a softening of the order parameter, presenting a weak order in the ordered phase. In the limit of Ising spins the fluctuating interaction renormalizes the mean field, reducing the critical intensity of the additive noise. [S1063-651X(97)10709-7]

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A noise-induced nonequilibrium phase transition in a dynamical system with multiplicative noise has been the topic of much recent investigation [1–11]. While an additive noise in equilibrium provides a disordering effect, restoring a broken symmetry, the multiplicative noise coupled to the state of the system induces an ordered symmetry-breaking state. The interplay of the additive and multiplicative noises produces a reentrant transition, showing the ordered symmetry-breaking state only for intermediate intensities of the multiplicative noise [4,11]. Most studies of the noise-induced phase transition have usually considered the fluctuating potential [4,6,8], explaining its mechanism by the short-time behavior of a single element [9]. Recently, the effect of fluctuating interaction has been studied showing the symmetry-breaking transition [7] and noise-enhanced multistability [10].

In this paper we investigate noise-induced phase transitions in soft Ising spins within the framework of the mean-field scheme. We introduce a fluctuating interaction into the system, showing that fluctuating interaction produces an effective ferromagnetic coupling and thus induces a ferromagnetic state even in the absence of ferromagnetic coupling. The interplay of the fluctuating interaction and the additive noise also leads to the reentrant transition. In the presence of ferromagnetic coupling the system shows a softening of the order parameter, leading to triple transitions. The softening implies the existence of a weak order in the ordered phase. In the limit of Ising spins the fluctuating interaction renormalizes the mean field, reducing the critical intensity of the additive noise.

A system of N coupled soft Ising spins under study is described by the equation of motion [12]

$$\frac{dx_i}{dt}=\mu(1-x_i^2)x_i-\frac{J}{z_i}\sum_j (x_i-x_j)+\sigma_A\xi_i(t), \tag{1}$$

where soft Ising spins x_i, $i=1, 2, \ldots, N$ are real variables; when $\mu\to\infty$, they asymptotically take on the values ± 1 leading to the Ising spins. The sum in Eq. (1) runs over z_i soft Ising spins coupled with the ith soft Ising spin, describing a ferromagnetic coupling which depends on the difference of two soft Ising spins. In Eq. (1), $\xi_i(t)$ is a Gaussian white noise characterized by

$$\langle \xi_i(t)\rangle=0,$$

$$\langle \xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t'),$$

where $\langle\ \rangle$ means an ensemble average over $\xi_i(t)$'s. σ_A measures the intensity of the additive noise $\xi_i(t)$ and is incorporated by the temperature $T\equiv\sigma_A^2/2$. Here we consider only the system with non-negative J.

Equation (1) is invariant under a symmetric operation

$$x_i\to -x_i, \quad \text{and} \quad \xi_i(t)\to -\xi_i(t). \tag{2}$$

In two- or higher-dimensional systems the inversion symmetry (2) is broken for small σ_A producing a ferromagnetic (FM) state characterized by a nonzero order parameter m defined by

$$m\equiv\frac{1}{N}\sum_i \langle x_i\rangle.$$

As σ_A increases, the equilibrium phase transition occurs at a critical value σ_{Ac} recovering the inversion symmetry (2) and producing a paramagnetic (PM) state with $m=0$. This is the well-known Ising-type phase transition.

To investigate effect of the fluctuating interaction, we introduce fluctuation into the ferromagnetic coupling J as

$$J\to J+\sigma_M\eta_i(t),$$

where $\eta_i(t)$ is a Gaussian white noise independent of $\xi_i(t)$, and characterized by

$$\langle \eta_i(t)\rangle=\langle \eta_i(t)\xi_j(t')\rangle=0,$$

$$\langle \eta_i(t)\eta_j(t')\rangle=\delta_{ij}\delta(t-t'),$$

with an ensemble average $\langle\ \rangle$ over ξ_i's and η_i's. σ_M measures the intensity of the fluctuating interaction providing a multiplicative noise. The fluctuating interaction does not break the symmetry [Eq. (2)] of the system. In the framework of the mean-field scheme, we replace the coupling term

*Electronic address: skim@logos.etri.re.kr

by the mean field m, obtaining the stochastic differential equation as a closed form of $x_i=x$,

$$\frac{dx}{dt}=\mu(1-x^2)x-[J+\sigma_M\eta(t)](x-m)+\sigma_A\xi(t). \quad (3)$$

This approach has been applied successfully in a number of other stochastic problems [4,13]

The macroscopic behavior of Eq. (3) can be described by the probability distribution $P(x,t)$ of x at time t, whose evolution is governed by the Fokker-Planck equation [14]

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\left[\mu(1-x^2)x-\left(J+\frac{\sigma_M^2}{2}\right)(x-m)\right]P(x,t)+\frac{\sigma_A^2+\sigma_M^2(x-m)^2}{2}\frac{\partial^2 P}{\partial x^2}. \quad (4)$$

In the steady state, Eq. (4) gives the stationary probability distribution

$$P(x)=\frac{1}{Z}\exp[-U(x)],$$

where $Z=\int_{-\infty}^{\infty}\exp[-U(x)]dx$, with

$$U(x)=2\mu U_\mu(x)+\left(\frac{J}{\sigma_M^2}+\frac{1}{2}\right)\ln[\sigma_A^2+\sigma_M^2(x-m)^2], \quad (5)$$

$$U_\mu(x)=-\frac{m}{\sigma_A\sigma_M}\left(1+3\frac{\sigma_A^2}{\sigma_M^2}-m^2\right)\tan^{-1}\left[\frac{\sigma_M}{\sigma_A}(x-m)\right]-\frac{1}{2\sigma_M^2}\left[1-3m^2+\frac{\sigma_A^2}{\sigma_M^2}\right]\ln[\sigma_A^2+\sigma_M^2(x-m)^2]+\frac{1}{2\sigma_M^2}(x-m)(x+5m). \quad (6)$$

Here m is given by the self-consistent equation

$$m=f(m)\equiv\int_{-\infty}^{\infty}xP(x)dx. \quad (7)$$

$U(x)$ plays the role of an effective potential in the system. We set $\mu=1$, representing σ_A^2, σ_M^2, and J in units of μ.

According to Ref. [4] the critical points at which the transition between the FM and PM states occurs are given by the roots of $f'(0)=1$, leading to

$$-\frac{1}{Z_0}\int_{-\infty}^{\infty}xU_1(x)\exp[-U_0(x)]dx=1, \quad (8)$$

where $Z_0=\int_{-\infty}^{\infty}\exp[-U_0(x)]dx$, with

$$U_0(x)=-\left[\frac{1-J}{\sigma_M^2}+\frac{\sigma_A^2}{\sigma_M^4}-\frac{1}{2}\right]\ln(\sigma_A^2+\sigma_M^2x^2)+\frac{x^2}{\sigma_M^2},$$

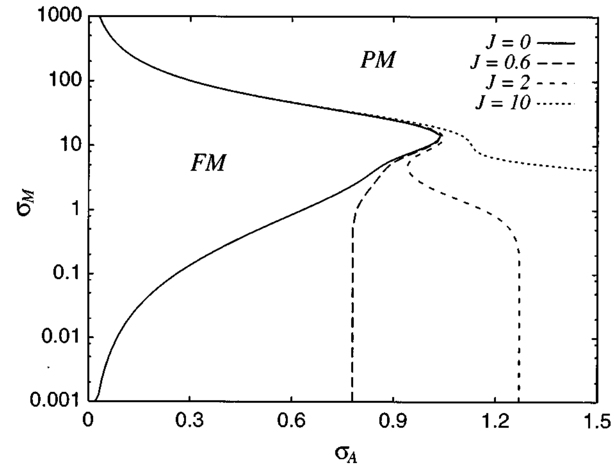

FIG. 1. Plots of phase diagram in the σ_A-σ_M plane for various values of J: solid line for $J=0$, long dashed line for $J=0.6$, short dashed line for $J=2$, and dotted line for $J=10$. PM and FM represent paramagnetic and ferromagnetic phases, respectively. σ_A and σ_M are in units of $\mu^{1/2}$, and J is in units of μ.

$$U_1(x)=-\frac{2}{\sigma_A\sigma_M}\left(1+3\frac{\sigma_A^2}{\sigma_M^2}\right)\tan^{-1}\left(\frac{\sigma_M}{\sigma_A}x\right)+2\left[1+\frac{\sigma_A^2}{\sigma_M^2}-J-\frac{\sigma_M^2}{2}\right]\frac{x}{\sigma_A^2+\sigma_M^2x^2}+\frac{4}{\sigma_M^2}x.$$

To obtain phase boundaries, we solve Eq. (8) numerically for various values of J.

Figure 1 shows the critical lines which separate the FM and PM phases in the σ_A-σ_M plane for various values of J. In the absence of the ferromagnetic coupling ($J=0$) the system shows double transitions as σ_M increases for small σ_A: When $\sigma_A=0$, the system is in the FM phase for all finite σ_M, implying that the fluctuating interaction induces the ferromagnetic order even in the absence of the ferromagnetic coupling. For $0<\sigma_A<\sigma_{Ac}\equiv1.04$, the system shows a reentrant transition, leading to two transition points σ_{Mc1} and σ_{Mc2} at which two transitions, PM→FM→PM, occur, respectively. As σ_A increases, σ_{Mc1} increases and σ_{Mc2} decreases. σ_{Mc1} and σ_{Mc2} meet at σ_{Ac}. For $\sigma_A>\sigma_{Ac}$ the system is in the PM state regardless of σ_M, implying that the disordering effect of the additive noise dominates the ordering effect of the fluctuating interaction entirely.

In the presence of the ferromagnetic coupling ($J>0$), the system without fluctuating interaction shows an equilibrium phase transition at a critical point $\sigma_{Ac}(\sigma_M=0)$. The phase transition persists as σ_M turns on, and increases up to some value of σ_M, σ_{Mc0}. When $J=0.6$, $\sigma_{Ac}(\sigma_M=0)$ and σ_{Mc0} are given by 0.78 and 0.53, respectively: As σ_M increases over σ_{Mc0}, $\sigma_{Ac}(\sigma_M)$ increases, and has a maximum value 1.04 at $\sigma_{Mc}\equiv15.1$. As σ_M increases further, $\sigma_{Ac}(\sigma_M)$ decreases approaching zero as σ_M goes to infinity. Thus, for $\sigma_A<0.78$, there is a single transition, FM→PM, as σ_M increases. For $0.78<\sigma_A<1.04$ the reentrant transition occurs, leading to the FM phase only in the intermediate values of σ_M. For $\sigma_A>1.04$ there is no phase transition showing the PM phase for all values of σ_M.

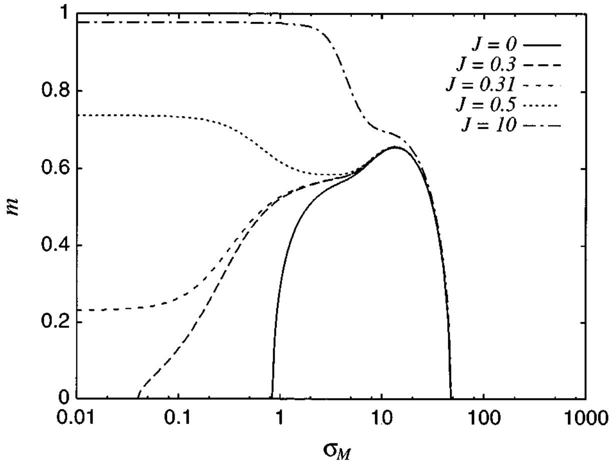

FIG. 2. Plots of the order parameter m as a function of σ_M for various values of J at $\sigma_A=0.6$: solid line for $J=0$, long dashed line for $J=0.3$, short dashed line for $J=0.31$, dotted line for $J=0.5$, and dash-dotted line for $J=10$. σ_A and σ_M are in units of $\mu^{1/2}$, and J is in units of μ.

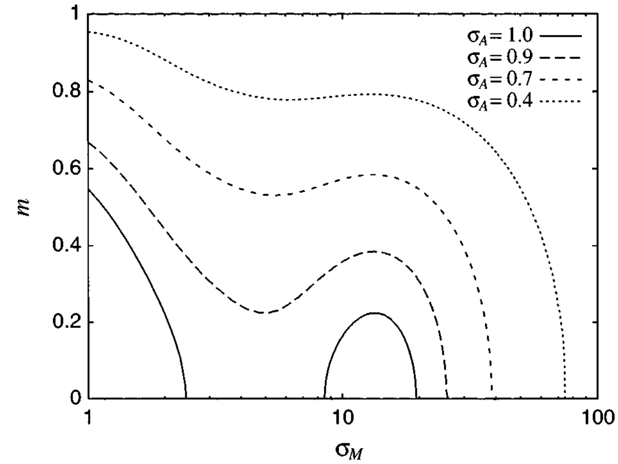

FIG. 3. Plots of the order parameter m as a function of σ_M for $J=2$ at various values of σ_A: solid line for $\sigma_A=1$, long dashed line for $\sigma_A=0.9$, short dashed line for $\sigma_A=0.7$, and dotted line for $\sigma_A=0.4$. σ_A and σ_M are in units of $\mu^{1/2}$, and J is in units of μ.

When $J=2$, the behavior of $\sigma_{Ac}(\sigma_M)$ is different with that for $J=0.6$: $\sigma_{Ac}(\sigma_M=0)$ and σ_{Mc0} are given by 1.27 and 0.22, respectively. As σ_M increases over σ_{Mc0}, $\sigma_{Ac}(\sigma_M)$ decreases, and has a minimum value 0.94 at $\sigma_{Mc1}=4.9$. As σ_M increase over σ_{Mc1}, $\sigma_{Ac}(\sigma_M)$ increases and has a maximum value 1.05 at $\sigma_{Mc2}=13.0$. As σ_M increases further, $\sigma_{Ac}(\sigma_M)$ decreases, approaching zero as σ_M goes to infinity. Thus for $\sigma_A<0.94$ or $1.05<\sigma_A<1.27$ there is a single transition, FM→PM, as σ_M increases. For $0.94<\sigma_A<1.05$, the triple transitions, FM→PM→FM→PM, occur as σ_M increases. For $\sigma_A>1.27$ there is no phase transition presenting the PM state for all σ_M. When J is large, a single transition FM→PM occurs as σ_M increases for all $\sigma_A<\sigma_{Ac}$, as shown in Fig. 1 with $J=10$.

Figure 2 shows the order parameter m, solution of the self-consistent equation (7), as a function of σ_M for various values of J at $\sigma_A=0.6$. When $J=0$, $m=0$ for $\sigma_M<\sigma_{Mc1}=0.85$; and as σ_M increases over σ_{Mc1}, m increases continuously, leading to the second-order phase transition PM→FM. At $\sigma_M=\sigma_{Mm}=11.5$, m has a maximum value 0.66, implying that the ordering effect of the fluctuating interaction is maximized. As σ_M increases further, m decreases, which means that the interplay of the additive and multiplicative noises reduces the ordering effect of the fluctuating interaction. At $\sigma_M=\sigma_{Mc2}\equiv 47.9$, m vanishes, leading to the reentrant transition FM→PM. While σ_{Mc1} decreases as J increases, σ_{Mc2} does not change for small J, as shown in Fig. 2. At some value of J, $J_c=0.305$, σ_{Mc1} vanishes, and above J_c the system with $\sigma_M=0$ is in the FM phase. When $J=0.5$, m has a dip, implying the existence of a weak order in the ordered phase. For large J, as shown in Fig. 2 with $J=10$, m decreases as σ_M increases, and vanishes at σ_{Mc2}, leading to a single transition FM→PM.

Figure 3 shows a softening of m for $J=2$, leading to the triple transitions at $\sigma_A=1$. For small σ_A, Fig. 3 shows a dip of m as σ_M increases: at $\sigma_M=0$, m has a nonzero value. As σ_M increases, m decreases, and has a minimum value $m_{\min}$ at σ_{Md}. As σ_M increases over σ_{Md}, m increases, and has a maximum value $m_{\max}$ at σ_{Mm}; and as σ_M increases over σ_{Mm}, m decreases vanishing at σ_{Mc}. The depth of the dip $m_{\min}$ decreases as σ_A increases and vanishes at a critical point σ_{Ac} leading to triple transitions. When $\sigma_A=1$, there are three transition points $\sigma_{Mc1}\equiv 2.5$, $\sigma_{Mc2}\equiv 8.5$, and $\sigma_{Mc3}\equiv 19.5$, at which the transitions FM→PM→FM→PM occur, respectively.

In the limit of $\mu\to\infty$, $U(x)$ given by Eq. (5) has two peaks at $x=\pm$, implying that x is restricted to ± 1. Since $U_\mu(1)\neq U_\mu(-1)$, in order to remove divergence we renormalize $U_\mu(x)$ by subtracting $F(x)$, characterized by

$$F(x)=\frac{U_\mu(-1)-U_\mu(1)}{32}[(x^3-3x)^3-12(x^3-3x)] + \frac{U_\mu(-1)+U_\mu(1)}{2} \tag{9}$$

near $x=\pm 1$. Since $F(\pm 1)=U_\mu(\pm 1)$ and $F'(\pm 1) = F''(\pm 1)=0$, it does not influence $U(x)$ up to the second order of $x\pm 1$ except for renormalization of $U(x)$. Using the steepest descent method in the limit of $\mu\to\infty$, $U(x)$ reduces to

$$U(s)=\frac{J}{\sigma_M^2}\ln[\sigma_A^2+\sigma_M^2(s-m)^2],$$

with an Ising spin $s\in\{-1,1\}$.

Neglecting a constant term, we transform $U(s)$ into

$$U(s)=-\frac{2JH(m)}{\sigma_A^2}s,$$

with an effective mean field $H(m)$ given by

$$H(m)=\frac{\sigma_A^2}{4\sigma_M^2}\ln\left[\frac{\sigma_A^2+\sigma_M^2(1+m)^2}{\sigma_A^2+\sigma_M^2(1-m)^2}\right].$$

Here $H(m)$ is a monotonic increasing function of m. In the limit of $\sigma_M \rightarrow 0$, $H(m)$ reduces to m, which is consistent with the mean field in the conventional Ising spin model. The self-consistent equation (7) leads to

$$m = \tanh\left[\frac{2J}{\sigma_A^2} H(m)\right],$$

producing the critical point

$$\frac{2J}{\sigma_A^2} H'(0) = \frac{2J}{\sigma_A^2 + \sigma_M^2} = 1. \tag{10}$$

The critical line (10) shows a circle centered at origin with radius $\sqrt{2J}$ in plane of σ_A-σ_M.

In this paper we investigated the effect of fluctuating interaction on the soft Ising spins within the framework of the mean-field scheme. Even in the absence of ferromagnetic coupling, the fluctuating interaction induces an ordered symmetry-breaking state, implying that the fluctuating interaction produces the effective ferromagnetic coupling. This noise-induced phase transition has different nature than that shown in Ref. [4]. In Ref. [4] the phase transition was induced by a fluctuating potential in the presence of a strong ferromagnetic coupling. The interplay of the additive noise and the fluctuating interaction induces the reentrant transition as shown in Ref. [4] and explained in Ref. [11].

In the presence of ferromagnetic coupling, the system shows various types of phase transitions. For small additive noise, the system shows a single transition, FM→PM, as σ_M increases. For intermediate additive noise, the system shows double transitions, PM→FM→PM, as σ_M increases. For some values of J and σ_A, the system shows triple transitions, FM→PM→FM→PM, as σ_M increases. The interplay of the ferromagnetic coupling, the additive noise, and the fluctuating interaction leads to a softening of the order-parameter m showing a dip as σ_M increases. The softening implies the existence of weak order in the ordered phase. We also obtained the renormalized mean field by the fluctuating interaction in the Ising spin limit. The renormalized mean field produces the critical line given by a circle centered at origin with radius $\sqrt{2J}$ in the σ_A-σ_M plane. We performed numerical simulations of a two-dimensional system with the fluctuating interaction, confirming that the mean-field approach has been applied successfully. The simulation results will be presented elsewhere.

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